原子核3者若手夏の学校

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Nonperturbative aspects of QFTs & Generalized Symmetries

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Goal
- strong dynamics of 4d gauge theories
- Power of (generalized) symmetries

Ref.
- 't Hooft や Witten の '70s ~ early '80s の論文.
- Aharony Seiberg Tachikawa '13, Gaiotto Kapustin Seiberg Willet '14
- Gaiotto Kapustin Komargodski Seiberg '17
- Nguyen, Tanizaki, Ünsal '23.

(Generalized Sym のより formal な話 ⇒ 昨日の大森さん)

Day1

約束: We focus on Euclidean QFTs in the path integral formalism.

$$Z = \int \mathcal{D}\phi \exp(-S[\phi])$$

$$\langle \phi(x_1)\phi(x_2)\rangle = \frac{1}{Z}\int\mathcal{D}\phi\, e^{-S}\phi(x_1)\phi(x_2)$$

With reasonable assumptions, we can reconstruct relativistic QFTs (Osterwalder-Schrader)

4d Yang-Mills theory について

$a = a_\mu dx^\mu$: $SU(N)$ gauge field. (gluon)

Gauge trans. $a \to a^U = U^{-1} a U - i U^{-1} dU$. (gluon is in the Adj. rep. $\square$)

(For $\phi \to U^{-1}\phi$, $(d+ia)\phi \to U^{-1}\cdot(d+ia)\cdot U \cdot U^{-1}\phi$)

Field strength :

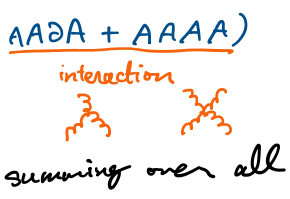
$$F = da + i a\wedge a \qquad (F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu + i[a_\mu, a_\nu])$$

transforms as $F \to F^U = U^{-1} F U$.

Lagrangian

$$\mathcal{L} = \underbrace{\frac{1}{g^2}\mathrm{tr}(F\wedge *F)}_{\frac{1}{g^2}\mathrm{tr}(F_{\mu\nu}F_{\mu\nu})d^4x}\left(+\, i\underbrace{\frac{\theta}{8\pi^2}\mathrm{tr}(F\wedge F)}_{\frac{\theta}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma}\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})d^4x}\right)$$

$$\sim \mathrm{tr}(\underbrace{\partial A\partial A}_{\text{propagator}} + \underbrace{AA\partial A + AAAA}_{\text{interaction}})$$

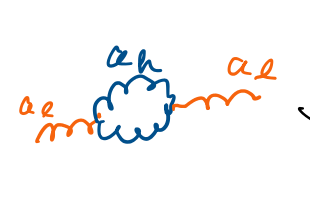


g^2: running coupling ; summing over all possible a_μ: a_l, a_h

$$Z = \int\mathcal{D}a\, e^{-S[a]} = \int \mathcal{D}a_{E<\mu}\mathcal{D}a_{E>\mu}\, e^{-S[a_l + a_h]}$$

$$= \int\mathcal{D}a_l\, e^{-S_{\mathrm{eff}}[a_l]}$$

$S_{\mathrm{eff}}[a_l]$: $\frac{1}{g^2_{\mathrm{eff}}}\mathrm{tr}[F_l\wedge *F_l] + \cdots$



$$g_{\mathrm{eff}} \sim g^2(\mu) \sim \frac{1}{\ln(\mu/\Lambda)}$$

Renormalization group

$$\Lambda = \mu\, e^{-\frac{8\pi^2}{\beta_0 g^2(\mu)}} \quad (\beta_0 = \tfrac{11}{3}N \text{ for pure YM})$$

RG scale

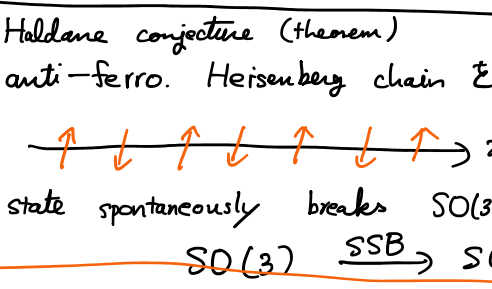
In QCD, $\Lambda \sim 1\,\mathrm{GeV} \sim \frac{1}{1\,\mathrm{fm}}$ typical mass / size of hadrons.

What we believe :
- Pure YM theory is gapped with the mass gap $\sim \Lambda$.
- colored d.o.f. are confined inside the hadrons of the size $\sim\Lambda^{-1}$.
  - quark confinement problem

⇒ We'd like to get better understandings on these nontrivial pheomena.

関連: Haldane conjecture (theorem)

(1+1)d anti-ferro. Heisenberg chain を考える. $H = J\sum_i \vec{S}_i\cdot\vec{S}_{i+1}$ $(J>0)$



(Néel state : classical vacuum)

Néel state spontaneously breaks SO(3) spin symmetry :

$$SO(3) \xrightarrow{SSB} SO(2) \quad \Rightarrow 2 \text{ Nambu-Goldstone modes.}$$

Low-energy theory is effectively described by 2d $\mathbb{CP}^1$ σ-model (at $\theta = 2\pi S$)

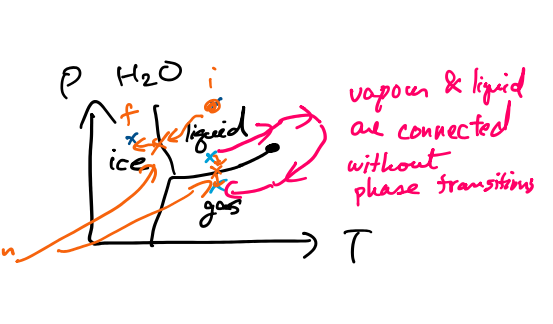
coset construction : Target space $\simeq \frac{SO(3)}{SO(2)} \simeq S^2 (\simeq \mathbb{CP}^1)$

This model is asymptotically free [Polyakov]

For $|x-y| \gtrsim \underbrace{a\, e^{\#\frac{1}{g^2}}}_{\sim\Lambda^{-1}} \gg a$ $\left\{\begin{array}{ll} \text{gapless} & \text{for } S\in\frac{1}{2}+\mathbb{Z}. \\ \text{gapped} & \text{for } S\in\mathbb{Z}. \end{array}\right.$ (Haldane gap) ($\sim S$)

Confinement とは?
- 新しい物質の相である.

Classical phases of matter :
- liquid & vapor belong to the same phase
- liquid & ice belong to different phases.

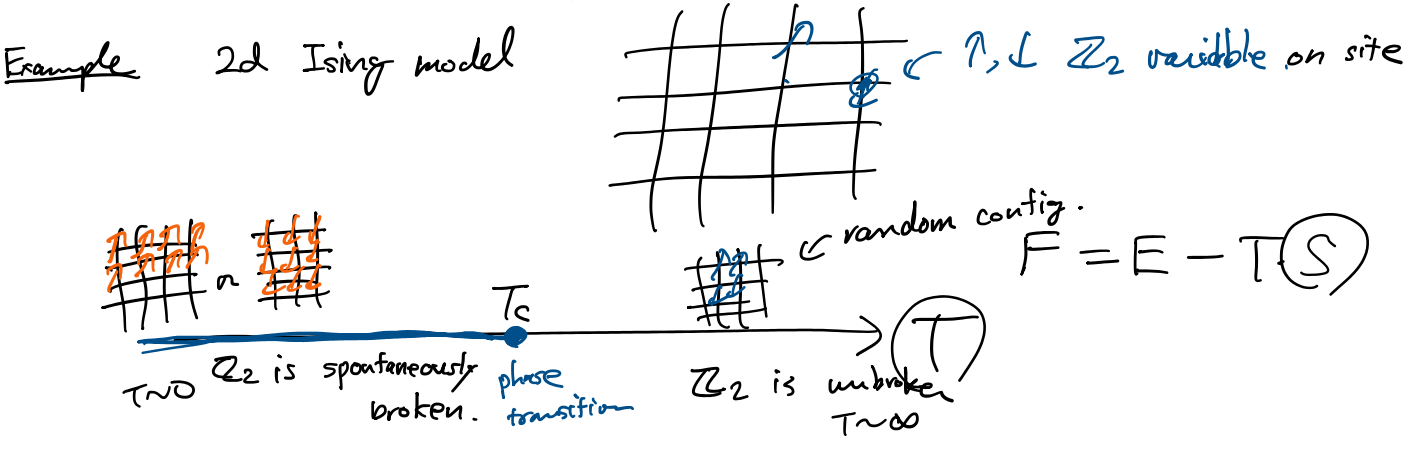


⇒ Landau criterion : Phases of matter can be diguished SSB pattern.

× Symmetry of system : translational invariance.

× Symmetry of each phase :
- × ice # $\mathbb{R}^3 \to (a\mathbb{Z})^3$: translational sym. is broken
- × liquid, × vapor } translati is unbroken.

Example 2d Ising model



↑,↓ $\mathbb{Z}_2$ variable on site
random config.
$F = E - TS$
T_c
$T\sim 0$ $\mathbb{Z}_2$ is spontaneously broken.
phase transition
$\mathbb{Z}_2$ is unbroken $T\sim\infty$

Can we use Landau criterion to classify quantum phases of matter?

- Gauge theory ⇒ · Topological matter. (topological order)
  - ex.) • Fractional Quantum Hall State.
  - • $\mathbb{Z}_2$ gauge theory (toric code)

  It seems that phases are separated without having symmetries.
  - · Symmetry-protected topological state.

⇒ Usual symmetries are not poweful enough to capture quantum phases.

⇒ We want to "generalize" the notion of global symmetries. generalized global symmetries

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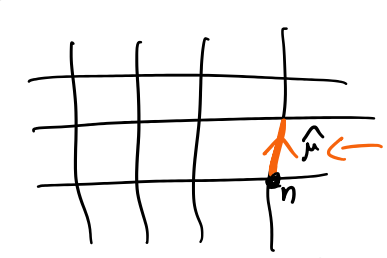
Examples for the phase str. of gauge theories

[Ref: Fradkin-Shenker]

とける confinement のモデル: Strong coupling expansion.



d.o.f. lives on link. $a = a_\mu dx^\mu$

$U_{n,\mu} = P\exp\left(i\int_n^{n+\hat\mu} a_\mu dx^\mu\right) \in G$: gauge group. (vector index)

$$U_{n,\mu} \xrightarrow{\text{gauge trans}} g(n+\hat\mu)^{-1}U_{n,\mu}\, g(n)$$

$$\sim \mathrm{tr}\left[\ \right] \sim \mathrm{tr}(F_{\mu\nu}F_{\mu\nu})$$

$$\to U_{n+\hat\nu,-\nu}U_{n+\hat\mu+\hat\nu,-\mu}U_{n+\hat\mu,\nu}U_{n,\mu} = U_{n,\mu\nu}$$

$$Z = \int\mathcal{D}U\, e^{-S_{\mathrm{lat}}[U]} \qquad \xi^{-1}: \text{correlation length for } G = SU(N)$$

$$S_{\mathrm{lat}} = \frac{1}{g^2}\sum_{n,\mu\nu}\mathrm{tr}(U_{n,\mu\nu}) \quad \leadsto \quad \Lambda \sim \frac{1}{a}e^{-\frac{8\pi^2}{\frac{11}{3}N g^2}}$$

strong coupling expansion $g^2 \gg 1$. [Wilson] [Creutz's textbook]

$$Z = \int\mathcal{D}U\, \underbrace{e^{+\frac{1}{g^2}\sum_n \mathrm{tr}(U_{n,\mu\nu})}}_{\prod_{n,\mu\nu}\left(1+\frac{1}{g^2}\mathrm{tr}(U_{n,\mu\nu}) + \cdots\right)}$$

Haar measure

$$= 1 + \sum_{\substack{\text{closed}\\\text{surface } \Sigma}}\left(\frac{1}{Ng^2}\right)^{\mathrm{Area}(\Sigma)}\left(\times N^{\chi(\Sigma)}\right) +$$

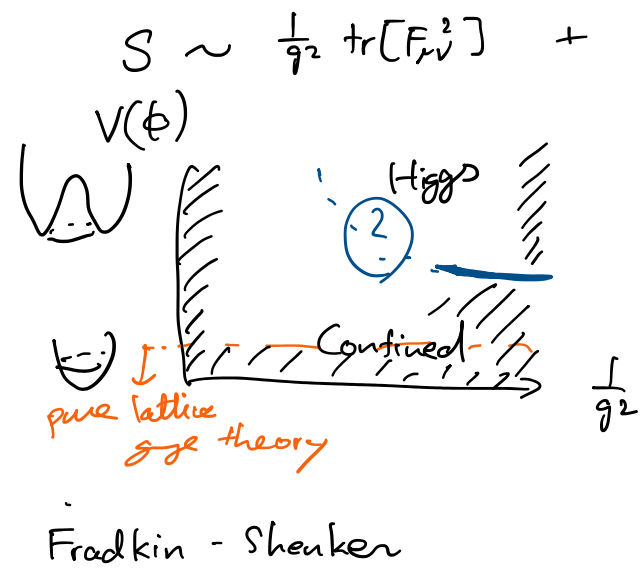
$$\sim \langle \phi_n \cdots\cdots \phi_{n+x} \rangle_{\mathrm{conn}}.$$

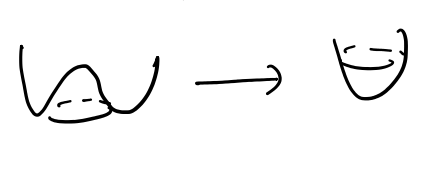
$$\sim \left(\frac{1}{g^2}\right)^{\#\cdot|x|} \sim e^{-|x|/\xi}$$

⇒ strong coupling limit is a confined theory

Lattice gauge + Scalar field

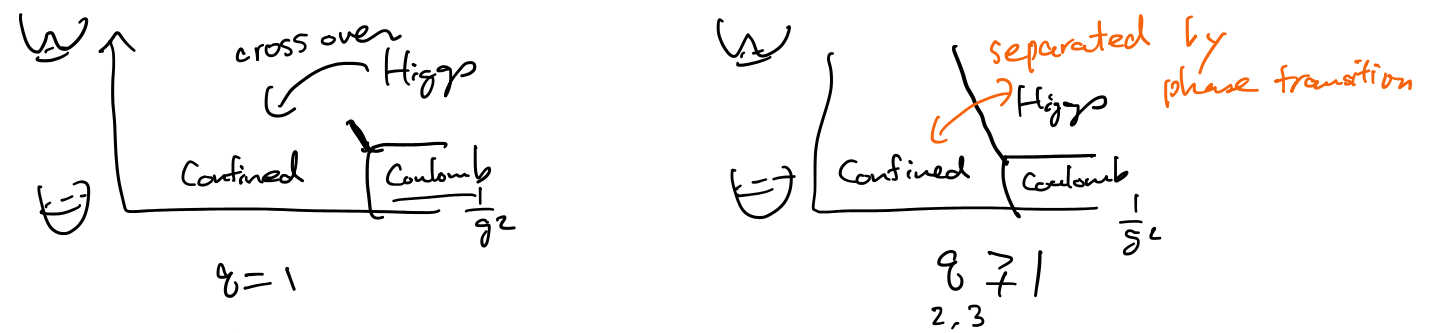
$$S \sim \frac{1}{g^2}\mathrm{tr}[F_{\mu\nu}^2] + |D_\mu\phi|^2 + V(|\phi|^2)$$



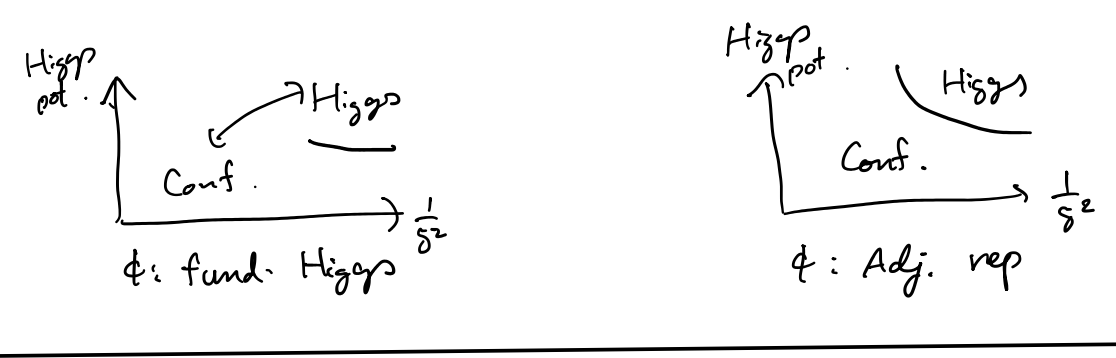


Fradkin-Shenker

① $G = U(1)$ φ: charged scalar. $D\phi = \partial\phi + i\,q\, a_\mu\phi$. (integer.)



② $G = SU(N)$



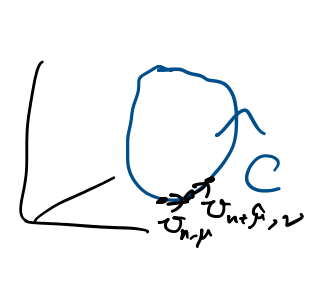
Day2

Order parameter for gauge theories : Loop operators. (esp. Wilson loop operator)

$$W(C) = \mathrm{tr}\left[P\exp\left(i\oint a\right)\right]$$

$$\left(\overset{\text{for lattice}}{=} \mathrm{tr}[\cdots U_{n+\hat\mu,\nu}U_{n,\mu}\cdots]\right)$$

path の順番に link をかけていく.

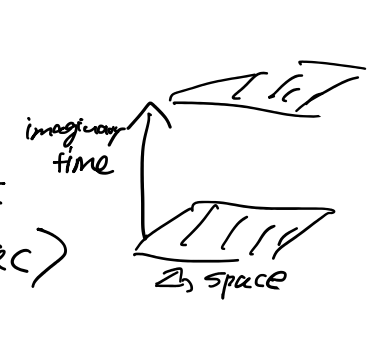


$$\left(\begin{array}{l}\mathrm{Det}[\gamma^\mu D_\mu + m] \\ \mathrm{Det}[-D_\mu D_\mu + m^2]^{-1}\end{array} = \sum_{\text{worldline } C} W(C)\right)$$

$$\langle W(C)\rangle = \frac{1}{Z}\int\mathcal{D}a\, e^{-S[a]}W(C)$$

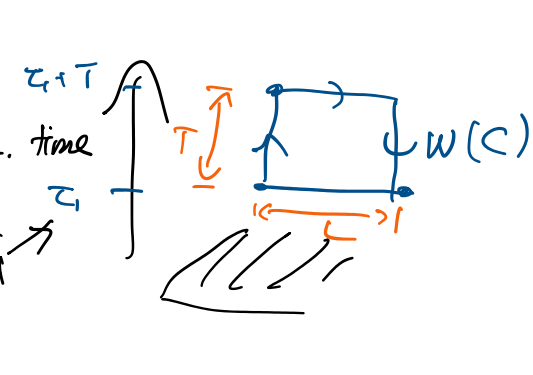
(Euclidean path int $\int \mathcal{D}x\, e^{-S[x]} = \langle x_f|e^{-iHt}|x_i\rangle$, $t \to -i\tau$

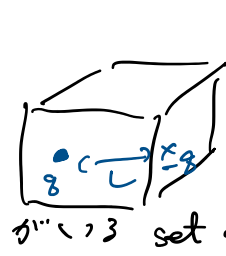
$$Z = \int\mathcal{D}\phi\, e^{-S[\phi]} = \langle\mathrm{vac}|e^{-TH}|\mathrm{vac}\rangle)$$



$$\langle W(C)\rangle = \langle\mathrm{vac}|e^{-T\hat{H}(q(x),\bar{q}(x+L))}|\mathrm{vac}\rangle$$

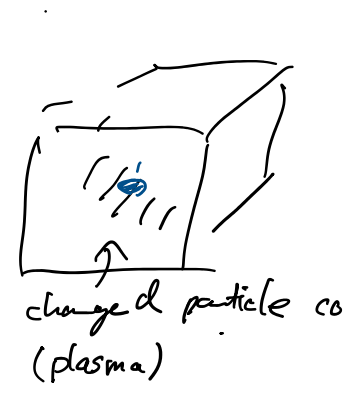
$$= e^{-T\cdot V(L)}$$



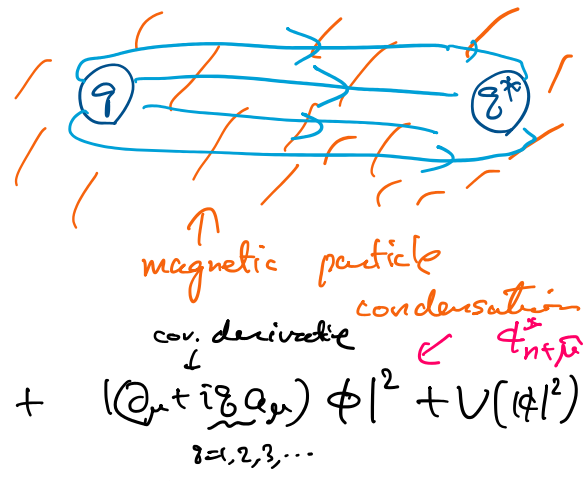


x に q がいて, $x+L$ に q^* がいる set up での minimal energy (= interparticle potential)

typical behaviors of V(L) :
- Coulomb-like behavior $V(L) \sim g^2\frac{1}{L}$ (← massless photon, gapless)
- Gapped state.
  - × Higgs phase $V(L) \overset{\text{Debye screening}}{\sim} \propto e^{-m\cdot L}$ (typical mass scale)
  - × Confinement phase (monopole condensation) $V(L) \sim \sigma\cdot L$ (string tension.)



charged particle condense. (plasma)



magnetic particle condensation

$U_{n,\mu}\in U(1)$

(Fradkin-Shenker.

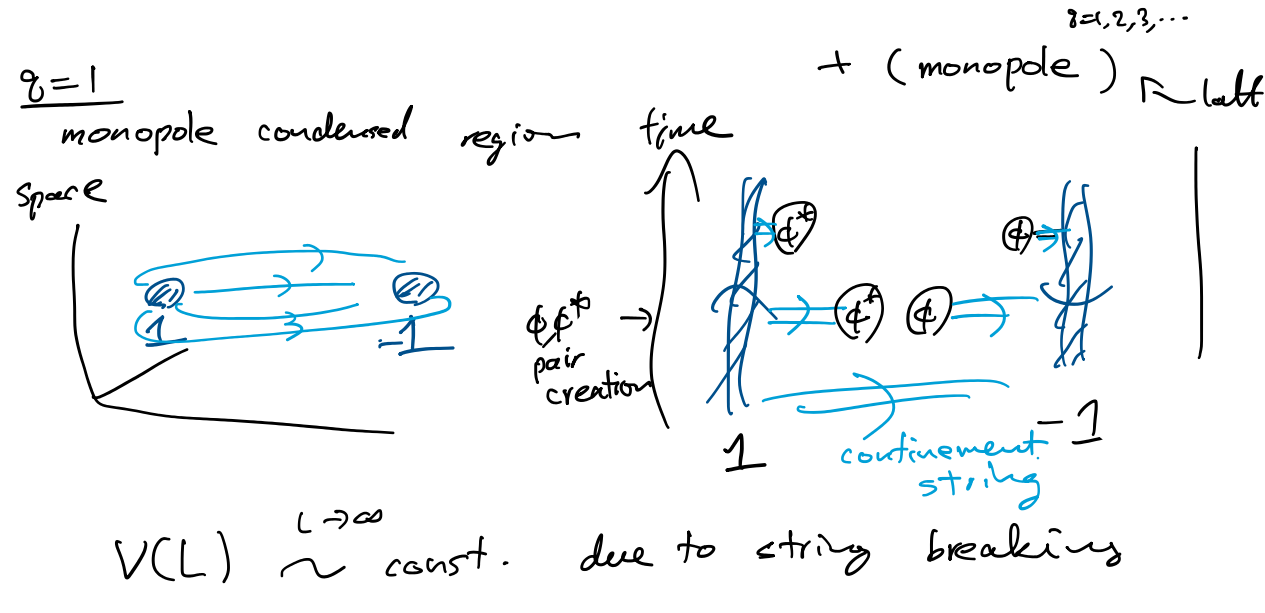
$$G = U(1), \text{ Higgs } S = \frac{1}{g^2}|da|^2 + |(\partial_\mu + iq a_\mu)\phi|^2 + V(|\phi|^2)$$

cov. derivative ; $q = 1,2,3,\cdots$; $\phi^*_{n+\hat\mu}(U_{n,\mu})^q\phi_n$

+ (monopole) ← lattice discretization

$q=1$

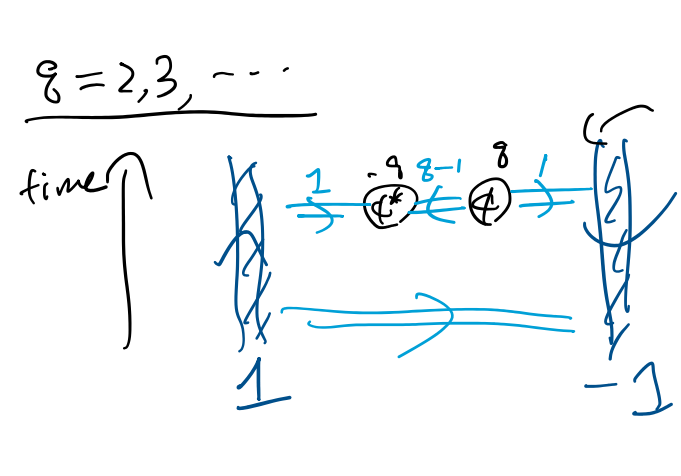
monopole condensed region



$$V(L) \overset{L\to\infty}{\sim} \text{const. due to string breaking}$$

↳ Although the system is confinement-like, interparticle pot. behaves in the same way in the Higgs region.

$q = 2,3,\cdots$



energetically unfavored.

$$\boxed{V(L) \sim \sigma\cdot L}$$

clearly distinguised from Higgs region.

We'd like to understand these phenomena in a more formal way.

- Generalized global symmetries : $\boxed{\text{1-form symmetry}}$

Motto

Generalized Symmetry = "Topological" operators in QFTs

Example

① Continuous symmetry

$$S[\phi] = S[e^{i\alpha}\phi] \quad \text{continuous rotation of fields.}$$

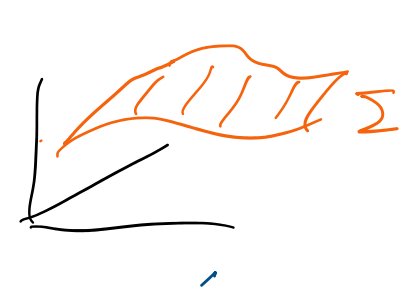
(α : const. $\to \alpha(x)$ space-time dependent

$$S[e^{i\alpha(x)}\phi(x)] = S[\phi] + \int d^dx\, j^\mu\partial_\mu\alpha + O(\alpha^2)$$

Noether current $\partial_\mu j^\mu = 0$ with EoM.

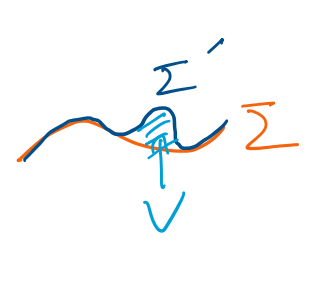
WT id. $\langle \partial_\mu j^\mu(x)\, \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = i\sum\delta(x-x_i)\langle\mathcal{O}_1\cdots\delta\mathcal{O}_i\cdots\mathcal{O}_n\rangle$

(⟺ As an "operator identity", $\partial_\mu j^\mu = 0$)



$$\int_\Sigma dS^\mu j_\mu = Q(\Sigma) \; : \text{Noether charge}$$

↳ Consider continuous deformati $\Sigma\to\Sigma'$,



$$Q(\Sigma') = Q(\Sigma) + \int_V d^dx\, \underbrace{\partial_\mu j^\mu}_{=0 \text{ (Noether cons.)}} \quad \leftarrow \text{Stokes thm}$$

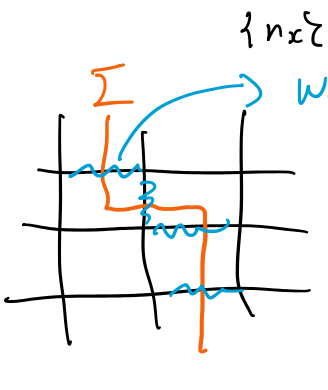
$$= Q(\Sigma)$$

↑ $Q(\Sigma)$ does not change under conti. deformati Σ.

⇒ $Q(\Sigma)$ is a topological operator.

② Ising model $S[n_x] = J\sum_{\langle x,x'\rangle} n_x n_{x'}$. $(n_x = \pm1)$

$$Z = \sum_{\{n_x\}} e^{-S[n_x]} \; : \mathbb{Z}_2 \text{ symmetry } n_x \to -n_x$$

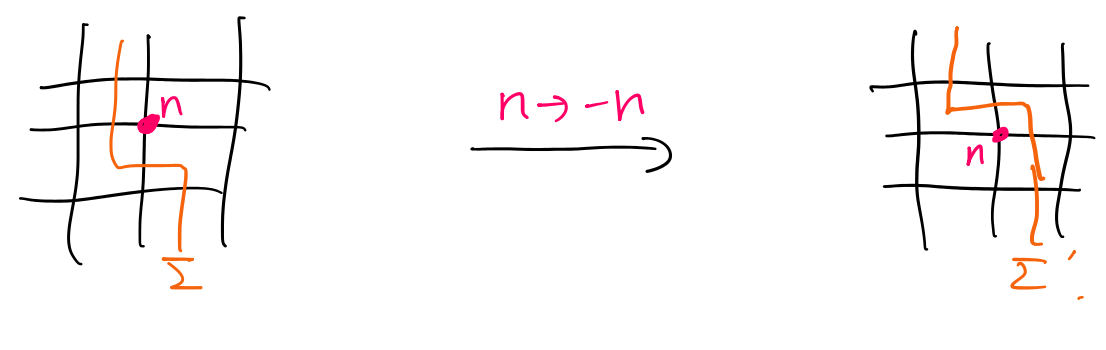


We change hopping ; $J \to -J$

$$\langle \underbrace{U(\Sigma)}_{\text{defect operator}} \cdots\rangle = \sum_{\{n_x\}} e^{-S_{\mathrm{twisted}}[J_{\langle x,x'\rangle}, n_x]}$$

$\left\{\begin{array}{l}+J\\-J\end{array}\right.$ usual

$U(\Sigma)$ is a topological operator :



We introduce $\mathbb{Z}_2$-link variables. ($U(\Sigma)$: background $\mathbb{Z}_2$ gauge field)

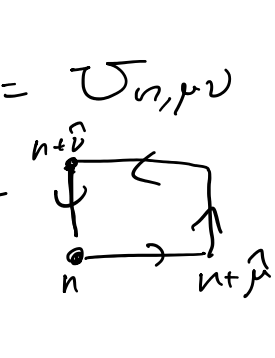
In this perspective, topologicalness of $U(\Sigma)$ is nothing but $\mathbb{Z}_2$-gauge invariance.

⇒ We can introduce the topological operator $\left(= \left\{\begin{array}{l}\text{Noether charge}\\\text{symmetry generator}\end{array}\right\}\right)$ by twisting the hopping term in the lattice model. (with symmetry)

③ (Electric) 1-form symmetry

Lattice gauge theory :

$$S = \sum_{\text{Plaquette}}\frac{1}{g^2}\mathrm{tr}[U_p] \qquad U_p = U_{n,\mu\nu} =$$



plaquette variables

$\left\{\begin{array}{lll} G = SU(N) & : \mathbb{Z}_N & \text{"1-form" symmetry} \\ G = U(1) & : U(1) & \text{"1-form" symmetry}\end{array}\right.$

↑ center group of the gauge group G. group consists of the elements that commute with all elements G.

$SU(N)$: $N\times N$ unitary matrix U w/ $\det(U) = 1$

$$Z(SU(N)) : x\mathbb{1}_N = \begin{pmatrix} x & & 0\\ & \ddots & \\ 0 & & x\end{pmatrix} \to x = e^{i\theta} \; ; \text{unitary}$$

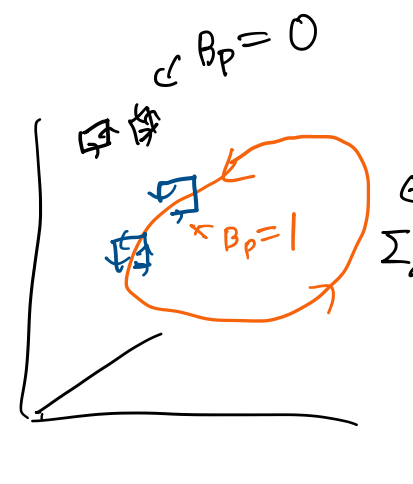
$$\det = x^N = e^{iN\theta} = 1 \; ; \text{special} \quad \theta = \frac{2\pi}{N}k.$$

$$\left\{e^{\frac{2\pi i}{N}k}\mathbb{1}_N \,\middle|\, k = 0,1,\ldots N-1\right\} \simeq \mathbb{Z}_N$$

Construction of top. defects :

$$S_{\mathrm{twist}}[\underbrace{e^{\frac{2\pi i}{N}B_p}}_{\text{twisting phase}}, \underbrace{U_l}_{\text{link variable}}] = \sum_p \frac{1}{g^2}e^{\frac{2\pi i}{N}B_p}\mathrm{tr}(U_p)$$

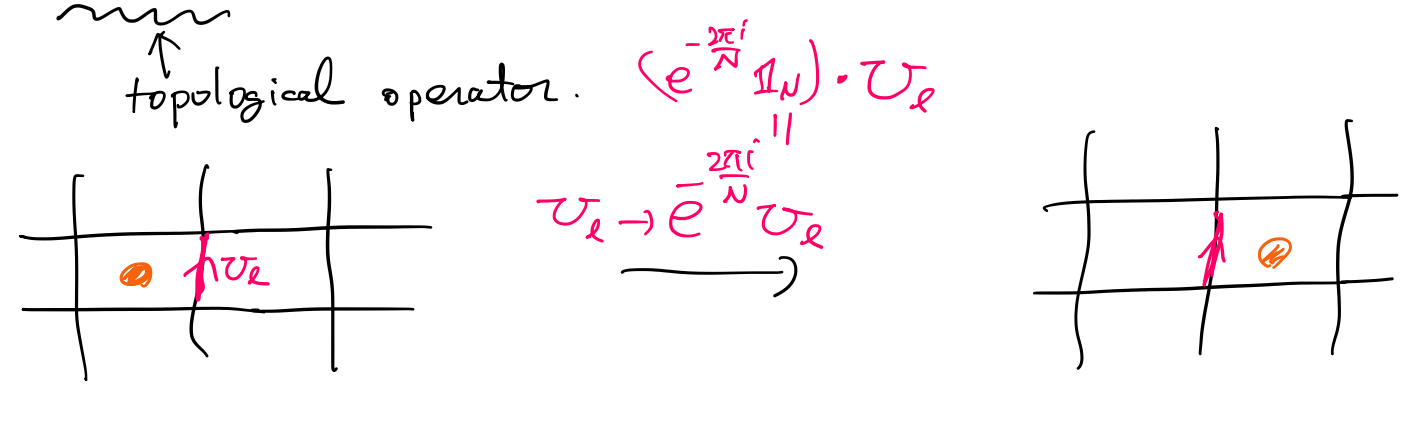
$P\prod_l U_l$; $\mathbb{Z}_N$ phase $e^{\frac{2\pi i}{N}B_p}$

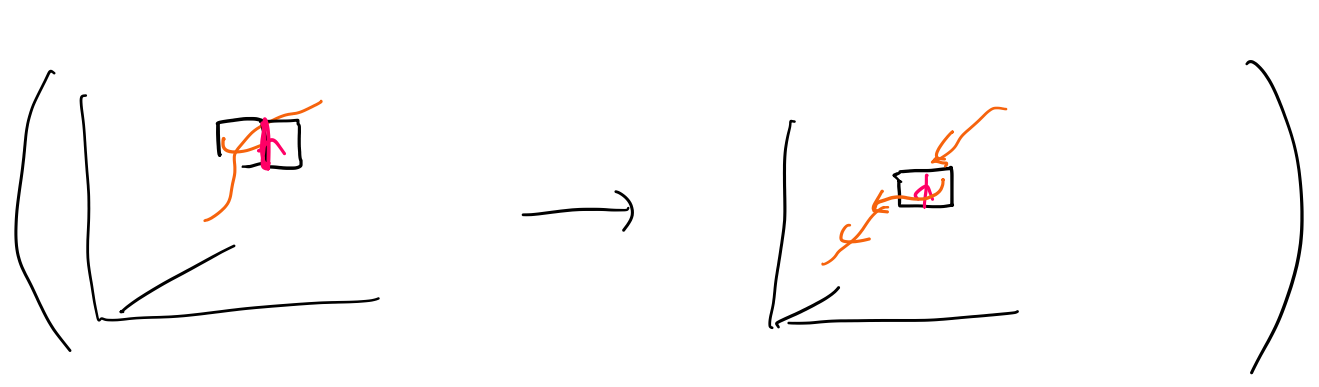


← codim-2 surface Σ_{d-2} (4d spacetime → surface, 3d ⋯ → line)

$$\langle \underbrace{U(\Sigma_{d-2})}_{\text{topological operator.}}\cdots\rangle = \int\mathcal{D}U_l\, e^{-S_{\mathrm{twist}}[B_p[\Sigma_{d-2}], U_l]} \cdots$$

$$\left(e^{-\frac{2\pi i}{N}}\mathbb{1}_N\right)\cdot U_l = e^{-\frac{2\pi i}{N}}U_l$$

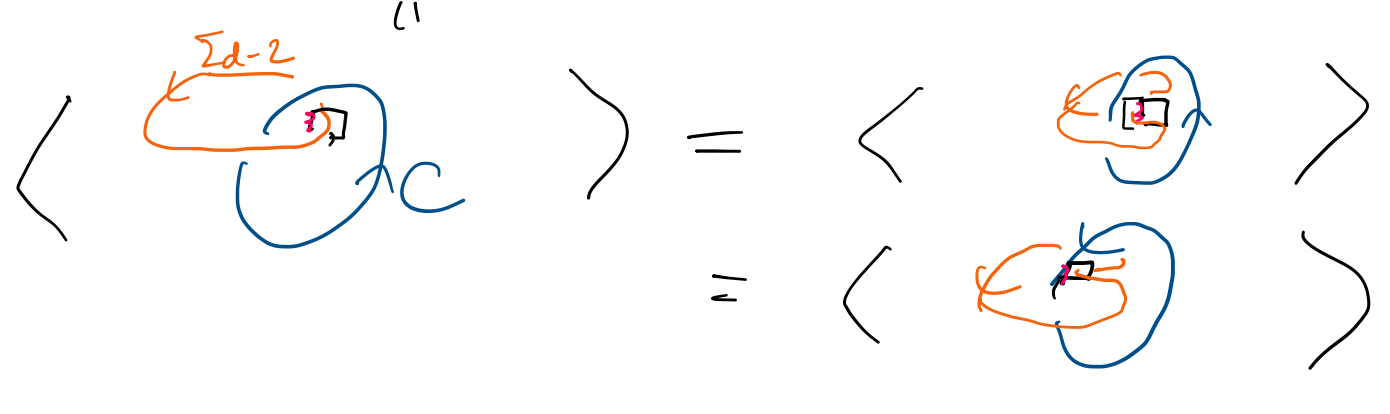




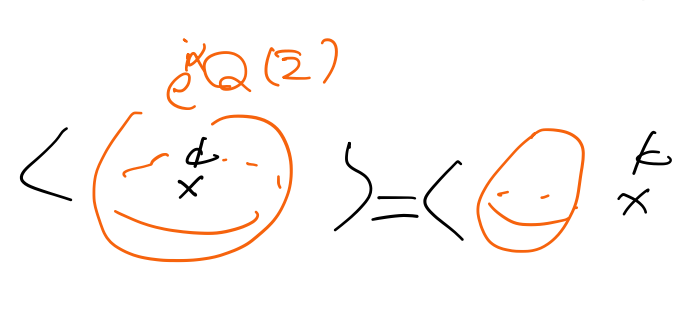
⇒ Pure gauge theory has a generalized symmetry generated by these codim-2 topological operators → 1-form symmetry (or center symmetry)

Charged operator : Wilson loop. $\mathrm{tr}[P\prod_{l\in C}U_l]$

$$\langle U(\Sigma_{d-2})\, W(C)\rangle$$



Σ_{d-2}
C
=
=
$= e^{\frac{2\pi i}{N}}\times$
$U_l \to e^{-\frac{2\pi i}{N}}U_l$



$$e^{i\alpha q_\phi}$$

1-form symmetry

$\left\{\begin{array}{lll}\text{Symmetry generator} & U(\Sigma_{d-2}) & ; \text{codim-2 top. defect}\\ \text{charged object} & W(C) & ; \text{Wilson loop}\end{array}\right.$

⇒ $\left\{\begin{array}{l}\bullet \text{ We can discuss SSB of 1-form symmetry.}\\ \bullet \text{ We can gauge 1-form symmetry.}\end{array}\right.$

Def. (SSB of (generalized) symmetries)

(Reminder : SSB of usual symmetry) $\left\{\begin{array}{l}\phi \to e^{i\alpha}\phi\\ \phi\to-\phi\end{array}\right.$ symmetric $S[\phi]$

(I) We consider Symmetry-breaking perturbation $\Rightarrow S[\phi] + \int \varepsilon\cdot\phi$

- ① Infinite volume limit. $\mathrm{Vol}\to\infty$ w/ $\varepsilon\neq0$
- ② Symmetry restorati limit $\varepsilon \to 0$

$$\lim_{\varepsilon\to0}\lim_{V\to\infty}\langle\phi\rangle_{V,\varepsilon} = v \quad \leftarrow \text{nonzero VEV} \Rightarrow \text{Symmetry is spontaneously broken}$$

(II) $\boxed{\text{ODLRO}}$ (Off-diagonal long-range order) ← We use this def. for SSB of generalized symmetry

(We don't need symmetry-breaking perturbation)

We compute connected 2-point func.

$$\langle\phi(x)\,\phi^*(y)\rangle_{\mathrm{conn.}} = \langle\phi(x)\phi^*(y)\rangle - \langle\phi(x)\rangle\langle\phi^*(y)\rangle$$

$$V\to\infty,\ |x-y|\to\infty \qquad = \langle\phi(x)\phi^*(y)\rangle$$

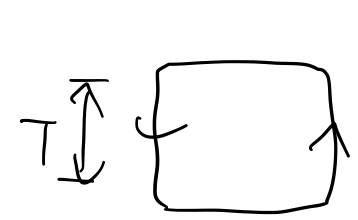
- Unbroken phase $\langle\phi(x)\phi^*(y)\rangle_{\mathrm{conn.}} \to 0.$
- Broken phase $\langle\phi(x)\phi^*(y)\rangle_{\mathrm{conn.}} \to v^2 \neq 0.$

↓

- SSB of 1-form symmetry

$$\langle W(C)\rangle \xrightarrow{|C|\to\infty} \left\{\begin{array}{ll} 0 & (\text{unbroken})\\ \neq 0 & (\text{broken})\end{array}\right.$$

(in a suitable renormalizati scheme)



$$\langle W(C)\rangle \sim e^{-T\cdot V(L) + \underset{\sim}{c}(T+L)}$$

renormalizati factor

Confinement phase

$$\langle W(C)\rangle \sim e^{-T\cdot\sigma L} = e^{-\sigma\cdot\overset{\mathrm{Area}(C)}{TL}} \; : \text{area law.}$$

⇒ 1-form sym is unbroken.

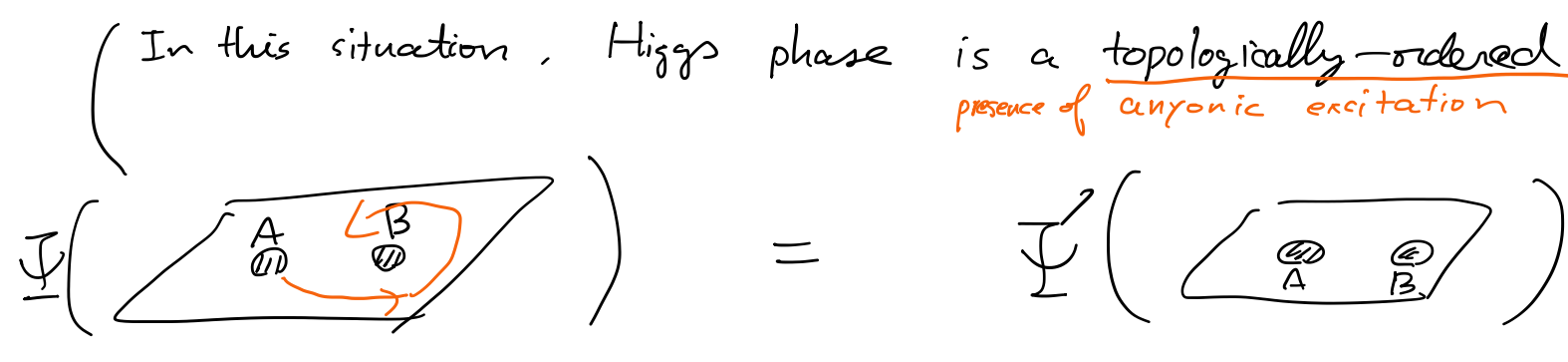
Higgs phase

$$\langle W(C)\rangle \sim e^{-\#\cdot(T+L) + c(T+L)} \underset{T,L\to\infty}{=} \text{const.}$$

(We choose the ren. factor

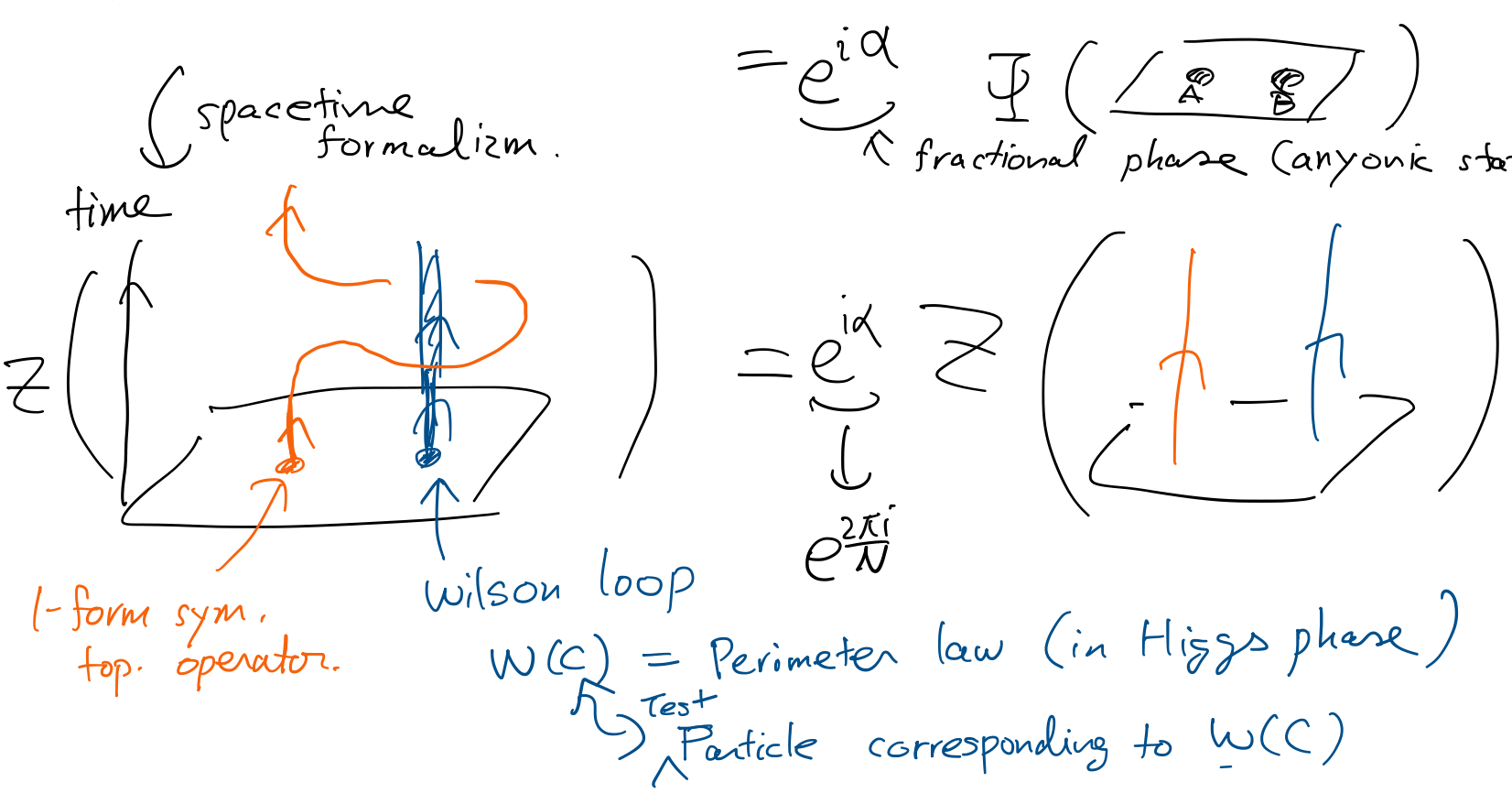
⇒ 1-form symmetry is spontaneously broken.

(In this situation, Higgs phase is a topologically-ordered phase (presence of anyonic excitation))



↖ fractional phase (anyonic statistics)

(spacetime formalizm.



1-form sym. top. operator.

Wilson loop

W(C) = Perimeter law (in Higgs phase)

↳ Test Particle corresponding to W(C) belong to low-energy states.

Fradkin-Shenker

$G = U(1)$, charge-q Higgs ; $\mathbb{Z}_q$ 1-form symmetry

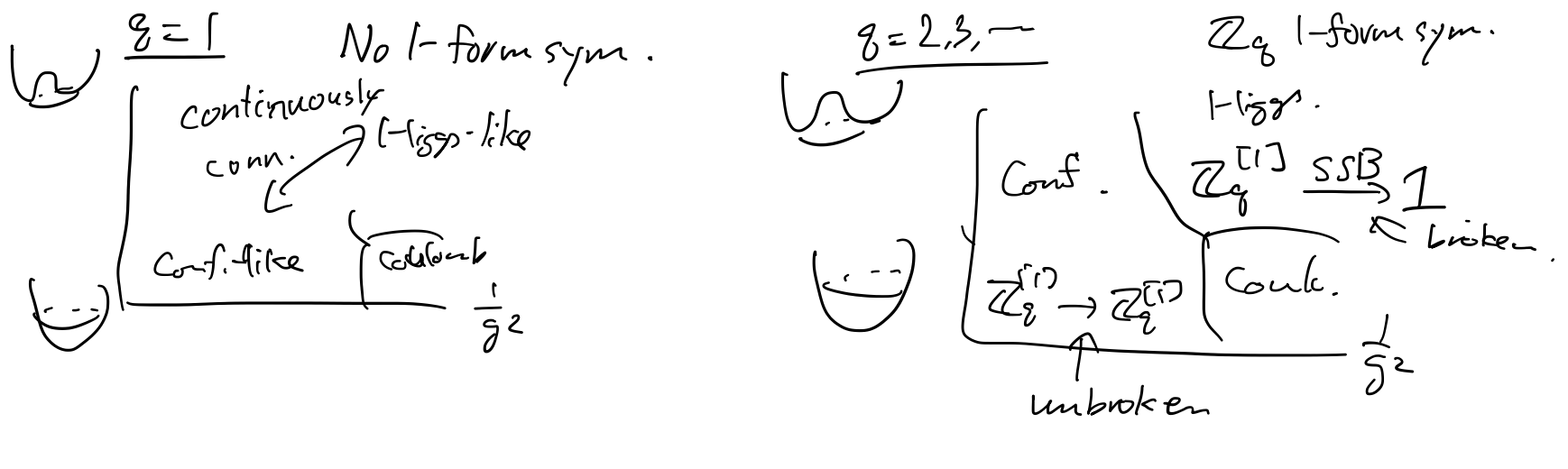
$$S_{\mathrm{twisted}} = \sum_p e^{iB_p}\frac{1}{g^2}U_p + \sum\left(\phi^*_{n+\hat\mu}(U_l)^q\phi_n + \mathrm{c.c.}\right)$$

$(U_l)^q \to e^{i\alpha q}\cdot(U_l)^q$

$U(1)$ 1-form symmetry $B_p\in[0,2\pi)$; $U_l \to e^{i\alpha}U_l$. $e^{i\alpha q} = 1$ is necersary for this operation to be symmetry.

$$e^{iB_p}\in\mathbb{Z}_q \quad \left(\text{i.e. } e^{iB_p}\in\left\{e^{i\frac{2\pi}{q}k}\right\}_{k=0,\ldots,q-1}\right)$$

$\left\{\begin{array}{l} q=1 \;:\; \text{There is no 1-form symmetry}\\ q=2,3,\cdots \;:\; \text{The model has } \mathbb{Z}_q \text{ 1-form symmetry}\end{array}\right.$



q=1
No 1-form sym.
continuously conn.
Higgs-like
Conf.-like
Coulomb
1/g²
q=2,3,⋯
$\mathbb{Z}_q$ 1-form sym.
Higgs.
Conf.
$\mathbb{Z}_q^{[1]} \xrightarrow{SSB} 1$
broken
$\mathbb{Z}_q^{[1]} \to \mathbb{Z}_q^{[1]}$
Coul.
unbroken
1/g²

We'd like to discuss the classification of phases more systematically.

$$Z = \int\mathcal{D}\phi\, e^{-S[\phi]} \to \langle\phi(x_1)\cdots\phi(x_n)\rangle = \int\mathcal{D}\phi\, e^{-S[\phi]}\phi(x_1)\cdots\phi(x_n)$$

→ Generating functional

$$Z[J] = \int\mathcal{D}\phi\, e^{-S[\phi] + \int\phi\cdot J}.$$

Computing $Z[J]$ is usually tough.

⇒ We focus on generating functional of Noether currents

$$Z[A_\mu] = \int\mathcal{D}\phi\, e^{-S[\phi] + i\int A_\mu J^\mu} \quad \leftarrow \text{Noether current}$$

(At least naively,

$$Z[A_\mu + \partial_\mu\theta] = \int\mathcal{D}\phi\, e^{-S[\phi] + i\int A_\mu J^\mu + i\int\partial_\mu\theta J^\mu}$$

$-\theta\cdot\partial_\mu J^\mu = 0$ ↗ int. by part

$$= Z[A_\mu] \;)$$

$A = A_\mu dx^\mu$: Background gauge field for global symmetry

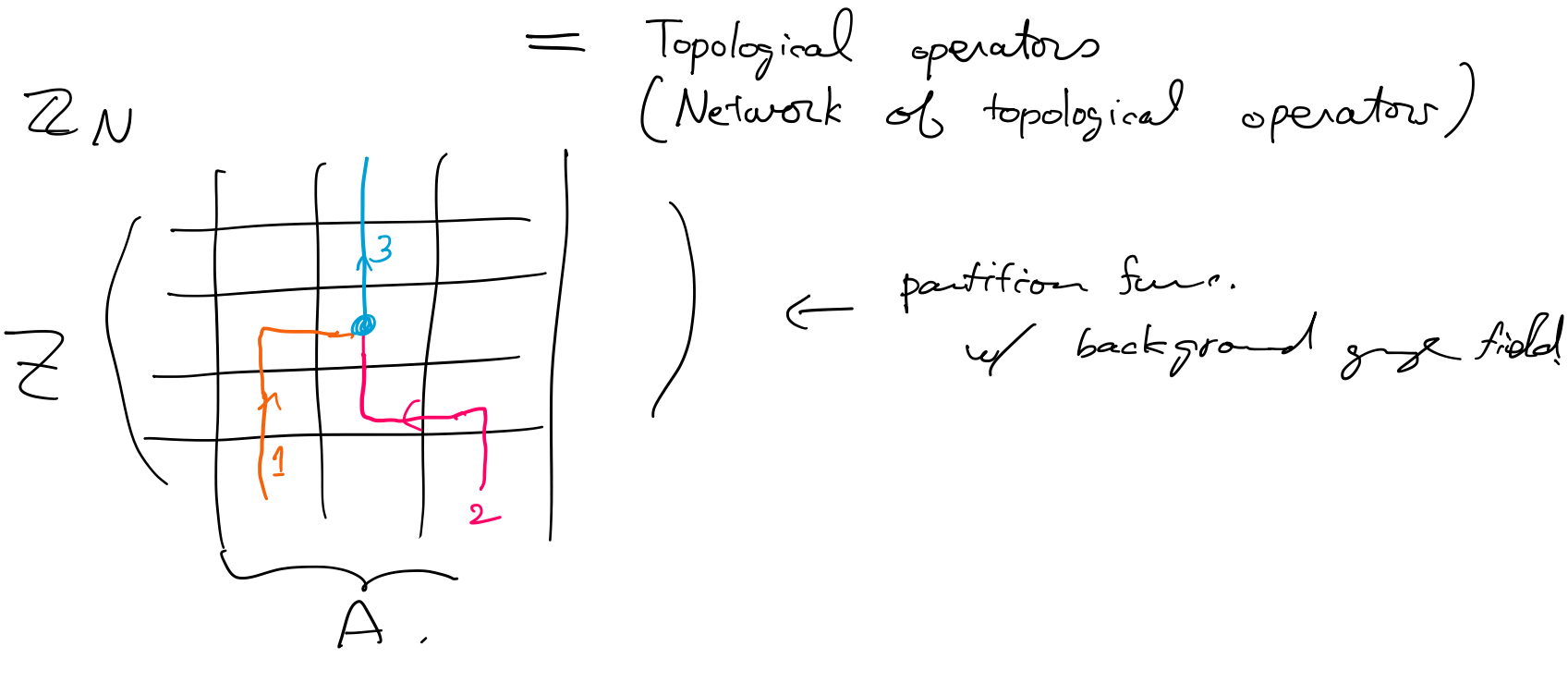
(In general, this background gauge inv. can be broken. ← 't Hooft anomaly

$$Z[A + d_A\theta] = e^{i\mathcal{A}[\theta,A]}Z[A] \;)$$

⇝ We can discuss the counterpart for generalized symmetries

Background gauge field A for generalized symmetries = Topological operators (Network of topological operators)

$\mathbb{Z}_N$



← partition func. w/ background gauge field

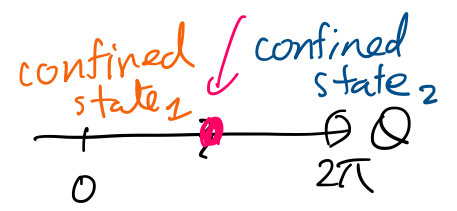
$Z[A]$ ← we can discuss various phases of matter by taking low-energy limit.

If symmetry is unbroken (& gapped)

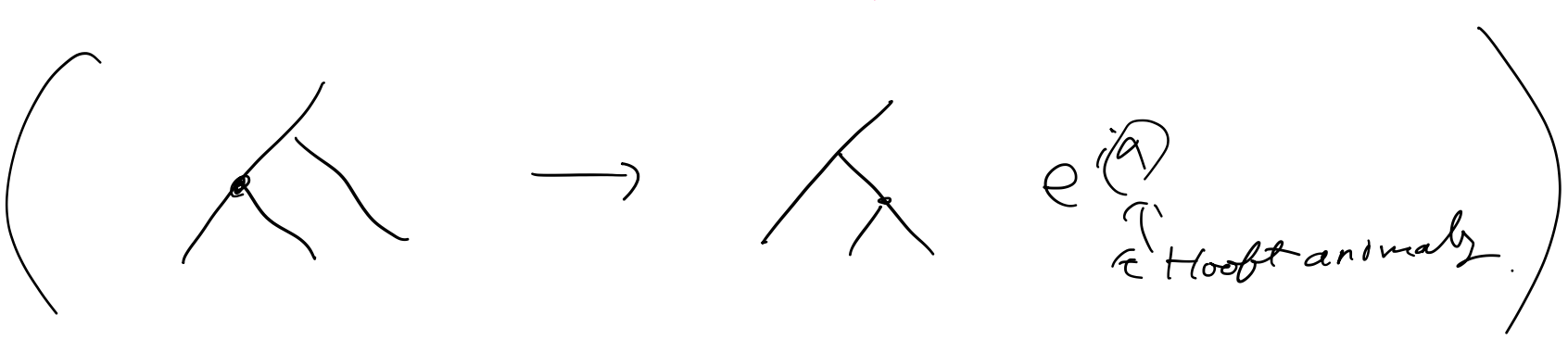
$$Z[A] \sim e^{iS_{\mathrm{eff}}[A]} \quad \leftarrow \text{"local" effective action of } A.$$

↑ In particular, if $S_{\mathrm{eff}}[A]$ contains the topological action of A, that is a new informati of quantum phases. (⟺ Symmetry-protected topological states)

YM example (Gaiotto, Kapustin, Komargodski, Seiberg '17)



$$Z_{\theta\sim0}[B] \sim 1 \qquad Z_{\theta\sim2\pi}[B] \sim e^{i\frac{N}{4\pi}\int B\wedge B}$$

different as $\mathbb{Z}_N^{[1]}$-protected SPT states.